SISSA 16/00/EP/FM
hep-th/0002210

A note on consistent anomalies in noncommutative YM theories

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Abstract. Via descent equations we derive formulas for consistent gauge anomalies in noncommutative Yang–Mills theories.

The growing interest in noncommutative YM theories is calling our attention upon old problems. We would like to know if or to what extent old problems, solutions or algorithms in commutative YM theories fit in the new noncommutative framework. One of these is the question of chiral anomalies . We would like to know what form chiral anomalies take in the new noncommutative setting. This has partially already been answered via a direct computation [1, 2]. In commutative YM theories there is another way to compute the explicit form of consistent chiral anomalies, i.e. by solving the Wess–Zumino consistency conditions (WZcc), [3], in particular via the descent equations, [4, 5, 6]. Now it is easy to see that in noncommutative YM theories WZcc still characterize chiral Ward identities, therefore one may wonder whether one can find the form of consistent anomalies by solving them, in particular whether a powerful algorithm like the descent equations is still at work. The answer is yes, as we briefly illustrate below.

In this note we have in mind a YM theory in a noncommutative $\mathbf{R}^D$, with Moyal deformation parameters θ^{ij}. We use the form notation for all the expression. Therefore we have a matrix–valued one–form gauge potential A, with gauge field strength two–form $F = dA + A * A$. We also introduce the gauge transformation parameter C, in the form of an anticommuting Faddeev–Popov ghost (using anticommuting gauge parameters simplifies a lot anomaly formulas, as is well–known). Next we introduce the following gauge transformation conventions:

$$\delta A = dC + A * C - C * A, \qquad \delta C = -C * C \tag{1}$$

The differential d is defined in the general non–commutative geometry setting as follows: it is the exterior derivative of the universal differential graded algebra $\Omega(\mathcal{B})$ associated to any algebra $\mathcal{B}$ [7, 8, 9]; $\mathcal{B}$ is for us the algebra generated by A and C. In simple words, this means that we deal with forms as usual, but never use the relation: $\omega_1\omega_2 = (-)^{k_1k_2}\omega_2\omega_1$, for any k_i–form ω_i.

d and δ are assumed to commute. As a consequence the transformations (1) are nilpotent as in the commutative case. They are noncommutative BRST transformations.

If one tries to derive for commutative YM theories descent equations similar to those of the commutative case, at first sight this seems to be impossible. In fact the standard expression one starts with, $\mathrm{Tr}(F \dots F)$, in the commutative case should be replaced by $\mathrm{Tr}(F * \dots * F)$ (Tr denotes throughout the paper the trace over matrix indices[1] ; but the latter is neither closed nor invariant, as one may easily realize. However one notices that it would be both closed and invariant if we were allowed to permute cyclically the terms under the trace symbol. In fact, terms differing by a cyclic permutation differ by a total derivative of the form $\theta^{ij}\partial_i$.... Such terms could of course be discarded upon integration. However, the spirit of the descent equations requires precisely to work with unintegrated objects.

The way out is then to define a bi–complex which does the right job. It is defined as follows. Consider the space of ($\mathcal{A}$–valued, where $\mathcal{A}$ is the algebra defining our non–commutative space) traces of $*$ products of such objects as A, dA, C, dC. The space of cochains is now this space, modulo the circular relation

$$\mathrm{Tr}(E_1 * E_2 * \dots * E_n) \approx \mathrm{Tr}(E_n * E_1 * \dots * E_{n-1})(-1)^{k_n(k_1+\dots+k_{n-1})} \tag{2}$$

where E_i is any of A, dA, C, dC, and k_i is the order form of E_i.

The definition of the bi–complex, let us call it $\mathcal{C}$, is completed by introducing two differential operators. The first is d, as defined above. The second differential is δ, the BRST cohomology operator. We define it to commute with d. Both preserve the relation (2).

We can now start the usual machinery of consistent anomalies, reducing the problem to a cohomological one. In a noncommutative even D–dimensional space we start with $\mathrm{Tr}(F * F * \dots * F)$ with n entries, $n = D/2 + 1$. In the complex $\mathcal{C}$ this expression is closed and BRST–invariant. Then it is easy to prove the descent equations:

$$\begin{aligned}
&\mathrm{Tr}(F * F * \dots * F) = d\Omega^0_{2n+1}\\
&\delta\Omega^0_{2n+1} = d\Omega^1_{2n}\\
&\delta\Omega^1_{2n} = d\Omega^2_{2n-1}
\end{aligned} \tag{3}$$

and so on. Here the Chern–Simons term can be represented in $\mathcal{C}$ by

$$\Omega^0_{2n+1} = n\int_0^1 dt\,\mathrm{Tr}(A * F_t * F_t * \dots * F_t) \tag{4}$$

where we have introduced a parameter t, $0 \leq t \leq 1$, and the traditional notation $F_t = t dA + t^2 A * A$.

The anomaly can instead be represented by

$$\begin{aligned}
\Omega^1_{2n} \;=\; & n\int_0^1 dt(t-1)\mathrm{Tr}(dC * A * F_t * \dots * F_t + dC * F_t * A * \dots * F_t +\\
& \dots + dC * F_t * F_t * \dots * A)
\end{aligned} \tag{5}$$

where the sum under the trace symbol includes $n-1$ terms.

[1]On more general noncommutative spaces (other than $\mathbf{R}^D$) this trace without an accompanying integration may not be well defined.

Finally

$$\Omega^2_{2n-1} = n\int_0^1 dt\frac{(t-1)^2}{2}\mathrm{Tr}(dC * dC * A * F_t * ... * F_t + \ldots)$$

where the dots represent $(n-1)(n-2)-1$ terms obtained from the first by permuting in all distinct ways dC, A and F_t, keeping track of the grading and keeping dC fixed in the first position.

The only trick to be used in proving the above formulas is to assemble terms in such a way as to form the combination $dA + 2tA * A = \frac{dF_t}{dt}$, and then integrate by parts.

In four dimensions the anomaly takes the form

$$\Omega^1_4 = -\frac{1}{2}\mathrm{Tr}(dC * A * dA + dC * dA * A + dC * A * A * A) \tag{6}$$

This anomaly, once it is integrated over, coincides with the result of [2], eq.(24) (modulo conventions).

On the basis of the above exercise, noncommutativity exhibits new qualitative features for chiral anomalies. Let us consider the case of $A = \sum_a A^a T^a$, where T^a is a basis of antihermitean matrices. The first two terms in (6) are proportional to $\mathrm{Tr}(T^aT^bT^c)$. Therefore the anomaly (6) vanishes only if $\mathrm{Tr}(T^aT^bT^c) = 0$, as was noticed in [2]. Now, $\mathrm{Tr}(T^aT^bT^c) = \frac{1}{2}\mathrm{Tr}(T^a\{T^b, T^c\}) + \frac{1}{2}\mathrm{Tr}(T^a[T^b, T^c])$. The first term in the RHS is the usual ad–invariant third order tensor; the second term, which is absent in the commutative case, is proportional to the structure constant and vanishes only when all the structure constants do. Analogous arguments apply to other dimensions. However the existence of a new part of the chiral anomaly that vanishes in the commutative case is of no use in the analysis of possible new cancellation mechanisms as long as the gauge group is $U(N)$, because the cancellation is driven by the $U(1)$ factor. In this case we reach the conclusion that noncommutative anomalies cannot vanish due to vanishing of ad–invariant tensors, as it occurs for many gauge groups in the commutative case: the only vanishing mechanism is therefore the one produced by matching anomaly coefficients with opposite chirality.

Note After this work was completed, J.Mickelsson informed us that the descent equations for chiral anomalies have been previously studied in [10]. The cohomology used in the two papers are different. Moreover the method used here is so much simpler, with results spelled out in detail, that we deem it worth a short note.

Acknowledgements We would like to thank C.P.Martin for a useful exchange of e-mail messages and L.Dabrowski, T. Krajewski and G.Landi for useful discussions. We thank J.Mickelsson for pointing out to us ref.[10]. This research was partially supported by EC TMR Programme, grant FMRX-CT96-0012, and by the Italian MURST for the program "Fisica Teorica delle Interazioni Fondamentali".

References

[1] F.Ardalan and N.Sadooghi, *Axial anomaly in noncommutative QED on* R^4, hepth/0002143.

[2] J.M.Gracia–Bondia and C.P.Martin, *Chiral gauge anomalies on noncommutative* R^4, hepth/0002171.

[3] J.Wess and B.Zumino, Phys.Lett. **B37** (1971) 95.

[4] R.Stora, in *New development in quantum field theories and statistical mechanics*, (eds. H.Levy and P.Mitter), New York 1977. and in *Recent progress in Gauge Theories*, ed. H.Letmann et al., New York 1984.

[5] L.Bonora and P.Cotta–Ramusino, Phys.Lett. **B107** (1981) 87. L.Bonora, Acta Phys. Pol. **B13** (1982) 799.

[6] B.Zumino, Wu Yong–shi and A.Zee, Nucl.Phys. **B239** (1984) 477.

[7] A.Connes, *Noncommutative geometry*, Academic Press, 1994.

[8] J.Madore, *An introduction to noncommutative differential geometry and its physical applications*, Cambridge University Press 1995.

[9] G.Landi, *An introduction to noncommutative spaces and their geometries*, Springer Verlag 1997.

[10] E.Langmann, *Descent equations of Yang–Mills anomalies in noncommutative geometry*, J.Geom.Phys. **22** (1997) 259-279.